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ON BILATERAL APPROXIMATION IN THE NUMERICAL INTEGRATION OF CAUCHY'S PROBLEM FOR ORDINARY DIFFERENTIAL EQUATIONS

by

S. Z. Zarubych

Dopovidi Akad. Nauk, Ukrain SSR, No. 10
PP 1278-1282, 1965.

Translated from the Ukrainian

June 1968

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ON BILATERAL APPROXIMATION IN THE NUMERICAL INTEGRATION OF CAUCHY'S PROBLEM FOR ORDINARY DIFFERENTIAL EQUATIONS

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The author constructs, in the manner of Gorbunov and Sahov [Z. Vycisl. Mat. i Mat. Fiz. 3 (1963, 239-253; MR 29 No. 739] and Devjatko [Ibid., 3 (1963), 254-265; MR 28 No. 5555], pairs of Runge-Kutta-Fehlberg formulas [Fehlberg, Z. Angew. Math. Mech. 38 (1958), 421-426; MR 20 No. 6791] having order five and principal error functions differing only in sign.

A method of bilateral approximations for the numerical integration of differential equations for Cauchy-type initial conditions was worked out by E. Ya. Remez [1], which makes it possible to easily evaluate the error introduced at any stage.

A number of works [1-4] have been devoted to the bilateral differential methods of Adams, Newton-Cotes, and Runge-Kutta, and there are here no difficulties in the choice of the upper and lower function, which is the case in solving differential equations by S. A. Chaplygin's method [5].

Let us construct a pair of formulas of the Runge-Kutta type such that the exact solution of a differential equation in a certain interval x_o, b, is contained between two approximate numerical solutions, one of which is below, whereas the other solution is above. In deriving formulas, use is being made of E. Fehlberg's transformations [6] and of the results of [3]. This makes it possible to construct bilateral formulas of a high degree of accuracy.

1. Differential Equations of the First Order

Let us consider a first-order differential equation

$$y' = f(x, y) \tag{1}$$

at

$$x = x_o \qquad y = y_o \,. \tag{2}$$

Instead of (1) and (2) we will examine the equation

$$z' = \varphi(x, z) \tag{3}$$

at

$$x = x_o \,, \qquad z = z_o = y_o \,. \tag{4}$$

At the point $x = x_o$

$$z'_o = 0 \,, \quad z''_o = 0 \,, \quad \left(\varphi_z\right)_o = 0 \tag{5}$$

and, therefore,

$$\left(\varphi_x\right)_o = 0 \,. \tag{6}$$

Conditions (3) through (6) obviously satisfy transformation [6]:

$$y = z + y'_o\left(x - x_o\right) + \frac{1}{2} y''_o\left(x - x_o\right)^2 + \left(f_y\right)_o\left(x - x_o\right)\left(z - z_o\right) \,. \tag{7}$$

For the right side of (3) we have

$$\varphi(x, z) = \frac{1}{1 + \left(f_y\right)_o\left(x - x_o\right)} \left\{ f\left(x, y(x, z)\right) - y'_o - y''_o\left(x - x_o\right) - \left(f_y\right)_o\left(z - z_o\right) \right\} . \tag{8}$$

We will present the accurate solution of problems (3) and (4) in the form:

$$z(x_1) = z_o + z'_o h + \frac{1}{2!} z''_o h^2 + \frac{1}{3!} z'''_o h^3 + \frac{1}{4!} z_o^{(4)} h^4 + \frac{1}{5!} z_o^{(5)} h^5$$

$$+ \frac{1}{6!} z_o^{(6)} h^6 + \frac{1}{7!} z_o^{(7)} h^7 + \ldots \tag{9}$$

by taking into account conditions (5), (6), and (3) we have:

$$z_o' = 0 \ , \qquad z_o'' = 0 \ , \qquad z_o''' = \varphi_{xx} \ , \qquad z_o^{(4)} = \varphi_{xxx} \ ,$$

$$z_o^{(5)} = \varphi_{xxxx} + 4\varphi_{xx}\varphi_{xz} \ ,$$

$$z_o^{(6)} = \varphi_{xxxxx} + 5\varphi_{xxx}\varphi_{xz} + 10\varphi_{xx}\varphi_{xxz} \ .$$

Let us consider the quantities:

$$k_1 = \varphi\left(x_o + \alpha_1 h, z_o\right)h \ ,$$

$$k_2 = \varphi\left(x_o + \alpha_2 h, z_o + \beta k_1\right)h \ ,$$

$$k_3 = \varphi\left(x_o + \alpha_3 h, z_o + \gamma k_1 + \delta k_2\right)h \ ;$$

$$k = z_1 - z_o = c_1 k_1 + c_2 k_2 + c_3 k_3 \ . \tag{10}$$

Having expanded the quantities $k_i (i = 1,2,3)$ in a Taylor series and denoting $\Delta z\left(x_o\right)$ by the exact value of the difference of solutions with Δz_o approximate, we conclude that $\Delta z\left(x_o\right) - \Delta z_o$ is a polynomial in h. Let the quantities $\alpha_i (i = 1,2,3)$, β, γ, δ, c $(i = 1,2,3)$ be chosen in such a way that

$$z(x_1) - z_1 = h^6 \alpha \mathcal{J}(\varphi)_o + 0(h^7) \ .$$

By selecting two values of α, differing only in sign, we obtain formulas of the bilateral method [3].

2. Derivation of Bilateral Methods Having Order Five of Accuracy

Let us consider the expression

$$\Delta z\ x_o - \Delta z_o = \frac{h^3}{2}\left[\frac{1}{3} - (\alpha_1^2 c_1 + \alpha_2^2 c_2 + \alpha_3^2 c_3)\right]\varphi_{xx}$$

$$+ \frac{h^4}{6}\left[\frac{1}{4} - (\alpha_1^3 c_1 + \alpha_2^3 c_2 + \alpha_3^3 c_3)\right]\varphi_{xxx}$$

$$+\frac{h^5}{24}\left[\frac{1}{5}-(\alpha_1^4c_1+\alpha_2^4c_2+\alpha_3^4c_3)\right]\varphi_{xxxx}$$

$$+\frac{h^5}{2}\left[\frac{1}{15}-(\alpha_1^2\alpha_2\beta c_2+\alpha_1^2\alpha_3\gamma c_3+\alpha_2^2\alpha_3\delta c_3)\right]\varphi_{xx}\varphi_{xz}$$

$$+\frac{h^6}{120}\left[\frac{1}{6}-(\alpha_1^5c_1\,\alpha_2^5c_2+\alpha_3^5c_3)\right]\varphi_{xxxxx}$$

$$+\frac{h^6}{4}\left[\frac{1}{18}-(\alpha_1^2\alpha_2^2\beta c_2+\alpha_1^2\alpha_3^2\gamma c_3+\alpha_2^2\alpha_3^2\delta c_3)\right]\varphi_{xx}\varphi_{xzz}$$

$$+\frac{h^6}{6}\left[\frac{1}{24}-(\alpha_1^3\alpha_2\beta c_2+\alpha_1^3\alpha_3\gamma c_3+\alpha_2^3\alpha_3\delta c_3)\right]\varphi_{xxx}\varphi_{xz}+\cdots \tag{11}$$

From (11) we will determine parameters so that

$$\Delta z\left(x_o\right)-\Delta z_o=h^6\alpha\psi(\varphi)_o+0(h^7)\ . \tag{12}$$

As a result we will obtain three pairs of formulas each of which is a bilateral Runge-Kutta type method for transformed equations (3) and (4):

$$k_1=\varphi\left(x_o+\frac{9}{10}h,z_o\right)h\ ,$$

$$k_2=\varphi\left(x_o=\frac{13}{15}h,z_o+\frac{2^3\cdot 13}{9^4}k_1\right)h\ ,$$

$$k_3=\varphi\left(x_o+h,z_o+\frac{10\cdot 3^2}{13^4}k_2-\frac{170\cdot 2^3}{3^5\cdot 11}k_1\right)h\ ,$$

$$k=z_1-z_o=-\frac{2\cdot 10^3}{3^5}k_1+\frac{3\cdot 15^3}{8\cdot 13^2}k_2+\frac{11}{8}k_3\ ,$$

$$x=x_o+h\ ,\quad z_1=z_o+k\ . \tag{13}$$

$$k_1=\varphi\left(x_o+\frac{9}{10}h,z_o\right)h\ ,$$

$$k_2=\varphi\left(x_o+\frac{1}{15}h,z_o+\frac{2\cdot 10^2}{9^4}k_1\right)h\ ,$$

$$k_3 = \varphi\left(x_o + h, z_o - \frac{43 \cdot 2^6}{3^5 \cdot 65} k_1 - \frac{154 \cdot 3^2}{65} k_2\right) h \ ,$$

$$k = z_1 - z_o = \frac{10 \cdot 2^4}{3^5} k_1 + \frac{5 \cdot 9^2}{56} k_2 - \frac{13}{56} k_3 \ ,$$

$$x = x_o + h \ , \qquad z_1 = z_o + k \ . \tag{13a}$$

$$k_1 = \varphi\left(x_o + \frac{2}{5} h, z_o\right) h \ ,$$

$$k_2 = \varphi\left(x_o + \frac{4}{5} h, z_o + \frac{4}{3} k_1\right) h \ ,$$

$$k_3 = \varphi\left(x_o + h, z_o + \frac{5}{128}(212k_1 - 45k_2)\right) h \ ,$$

$$k = z_1 - z_o = \frac{1}{1152}(500k_1 + 375k_2 + 64k_3) \ ,$$

$$x_1 = x_o + h \ , \qquad z_1 = z_o + k \ . \tag{14}$$

$$k_1 = \varphi\left(x_o + \frac{2}{5} h, z_o\right) h \ ,$$

$$k_2 = \varphi\left(x_o + \frac{4}{5} h, z_o + \frac{4}{3} k_1\right) h \ ,$$

$$k_3 = \varphi\left(x_o + h, z_o + \frac{5}{128}(99k_2 - 364k_1)\right) h \ .$$

$$k = z_1 - z_o = \frac{1}{1152}(500k_1 + 375k_2 + 64k_3) \ ,$$

$$x_1 = x_o + h \ , \qquad z_1 = z_o + k \ . \tag{14a}$$

The solution of problems (3) and (4) is found as the mean value of the sum of the upper and lower approximation for $z(x)$. By substituting the values found into equality (7) we will find the solution of problems (1) and (2).

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